ON PERFECTLY PLASTIC FLOW OF MATERIAL WITH RESIDUAL MICROSTRESSES TAKEN INTO ACCOUNT

(OB IDEAL'NO PLASTICHESKOM TECHENII MATERIALA S UCHETOM OSTATOCHNYKH MIKRONAPRIAZHENII)

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In [1], Novozhilov and Kadashevich proposed a two-dimensional dynamical model which facilitates the description of behavior of an anisotropic strain-hardening material [2,3]. The ideas of [1] can be utilized for construction of a large class of models of continuous media.

On the basis of an appropriate two-dimensional dynamical model, we shall consider the relations determining perfectly plastic flow of a material in which (according to the terminology of Novozhilov and Kadashevich) residual microstresses develop.

1. For a group of materials, the relation between the shear stress τ and the shear deformation γ can be adequately described by a diagram shown in Fig. 1 (in the following, only rigid-plastic materials will be considered). It is essential that, in this case, the yield stress $\tau = k_1$ (point A in Fig. 1) and the flow stress $\tau = m$ (point B in Fig. 1) do not coincide. The segment AB ($k_1 \leqslant \tau < m$) characterizes hardening of the material and, in general, it is nonlinear. At $\tau = m$, a perfectly plastic flow develops.

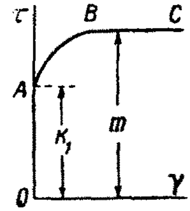

Fig. 1.

We shall consider the behavior of the material in the perfectly plastic flow (segment BC in Fig. 1). In our case, not as in the usual formulation of the theory of rigid-perfectly-plastic solids [4], several different approaches to the construction of the theory are possible.

If the hardening solid remains isotropic, the relations of the perfectly plastic flow have the usual form [4]. In the case of anisotropic hardening, the characteristic properties of perfectly plastic flow are influenced by the arising residual microstresses. For simplicity, we shall consider the case of an ideal Bauschinger effect.

We shall consider first a one-dimensional model (Fig. 2*a*) consisting of two elements with solid friction connected with a spring. Denoting by k_1 and k_2 the limiting frictional forces of the first and the second element, respectively, the relation between the traction T and the displacement q is given in Fig. 2*b*. The nonlinearity on the segment AB can be obtained by assuming a nonlinear characteristic of the spring. The corresponding two-dimensional model is shown in Fig. 3.

It is obvious that the behavior of the models is essentially influenced by the forces in the springs aa_1 and bb_1 (Fig. 3). These forces correspond to microstresses in a continuous solid. The case in which the forces in the springs aa_1 and bb_1 do not exceed the force of solid friction in the element 2 has been considered in [1]. In this paper, we shall consider a perfectly plastic flow at which the forces in the springs aa_1 and bb_1 are large enough to overcome the resistance of solid friction in the element 2.

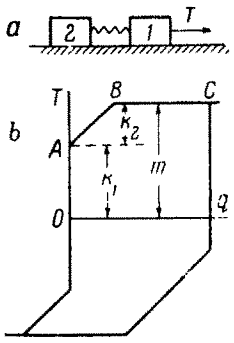

Fig. 2.

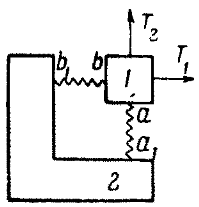

Fig. 3.

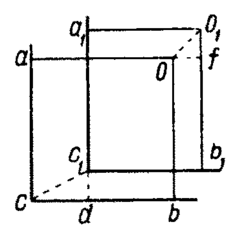

Fig. 4.

We denote by T_1 and T_2 the external forces, and by s_1 and s_2, the forces in the springs aa_1 and bb_1. We assume that the forces T_i and s_i are such that the elements *1* and *2* acquire increments of displacement. In Fig. 4, the initial position of the element *1* is represented by the point o and the subsequent position, by the point o_1. The element *2* moves from the position acb into the position $a_1b_1c_1$. The following conditions should be satisfied:

$$(T_1 - s_1)^2 + (T_2 - s_2)^2 = k_1^2 \tag{1.1}$$

$$s_1^2 + s_2^2 = k_2^2 \tag{1.2}$$

We denote now the displacement of the element *1* by q_1, q_2 and the displacement of the element 2 by r_1, r_2. From Fig. 4 we have $of = \Delta q_1$ $o_1 f = \Delta q_2$, $cd = \Delta r_1$, $c_1 d = \Delta r_2$.

The elastic constants of the springs aa_1 and bb_1 will be denoted by $1/c$ and, thus,

$$\Delta(q_1 - r_1) = c\Delta s_1, \qquad \Delta(q_2 - r_2) = c\Delta s_2 \tag{1.3}$$

The displacements of the elements *1* and *2* coincide with the directions of acting forces; therefore

$$\frac{\Delta q_1}{\Delta q_2} = \frac{T_1 - s_1}{T_2 - s_2}, \qquad \frac{\Delta r_1}{\Delta r_2} = \frac{s_1}{s_2} \tag{1.4}$$

The relations (1.1) - (1.4) allow for investigation of the behavior of the system shown in Fig. 4. It is necessary to point out the qualitative character of these considerations and to warn against any far going analogies between the behavior of the dynamical model and the behavior of a continuous medium. We do not consider, for instance, rotations of the element 2, and related effects, as they are immaterial for the following discussion.

Certain aspects of perfectly plastic flow can be conveniently illustrated, in this case, with kinematical models [5]. For an anisotropic hardening material with the ideal Bauschinger effect, a model can be used in the form of a circular frame moving in the plane under the action of the pivot A if there is no friction between the pivot and the frame (Fig. 5). The point O represents the initial center of the frame and the point O_1, a subsequent center of the frame. The distance AO corresponds to stresses, and the distance OO_1 corresponds to deformations.

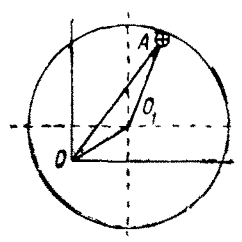

Fig. 5.

In our case, the corresponding kinematical model is the following. We take two circles which are concentric in the initial state. One of them is moving under the action of the pivot A. This same circle has in its center the pivot O_1 acting upon the second circle. There is no friction between the pivots and the circles. The point O is the initial center of both circles (corresponding to the natural state), the point O_1 is the center of the first circle, and the point O_2 is the center of the second circle (Fig. 6).

The distance AO_2 corresponds to stresses, O_1O_2 corresponds to microstresses, and OO_1 corresponds to deformations.

We note that at a developed flow whose loading path approaches a straight line, the points A, O_1, O_2 tend to occupy positions also on a straight line.

2. Using dynamical analogies we interpret forces as stresses and displacements as deformations.

We denote by σ_{ij} the tensor of actual stresses (corresponding to the forces T_i) by s_{ij} the tensor of microstresses (corresponding to the forces t_i), by e_{ij} the tensor of actual strains (corresponding to the displacements q_i of the element *1*), and by κ_{ij} the tensor of internal microstrains (corresponding to the displacements r_i of the element *2*). Deviators of these tensors are denoted by dashes above the symbols.

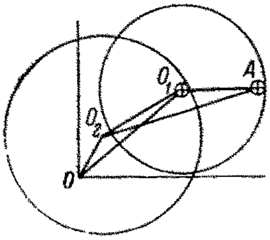

Fig. 6.

The components of the stress tensor σ_{ij} satisfy the equations of equilibrium $\sigma_{ij,j} = 0$; the components of the strain tensor e_{ij} can be expressed in terms of the components of displacement $e_{ij} = 1/2(u_{i,j} + u_{j,i})$.

The behavior of the material in the case of microstresses s_{ij} which are unable to exceed the resistance of solid friction in the corresponding elements (the element *2* in Fig. 2*a*) is described by the equations of the theory of anisotropic hardening and it has been investigated in the papers [1-3] and others. Here, we shall consider the equations of the perfectly plastic flow; the stresses σ_{ij} and s_{ij} being such that resistance of solid friction in both elements is overcome.

We write the flow conditions in the form

$$f_1(\sigma_{ij} - s_{ij}) = k_1, \qquad f_2(s_{ij}) = k_2 \tag{2.1}$$

Consider now the expressions for the increments of the works

$$dA_1 = \sigma_{ij} de_{ij}, \qquad dA_2 = s_{ij} d\varkappa_{ij} \tag{2.2}$$

Assuming that the expressions dA_1 and dA_2 are stationary at the respective conditions (2.1), we determine the flow rule considering the expressions (2.1) as plastic potentials

$$de_{ij} = d\lambda_1 \frac{\partial f_1}{\partial \sigma_{ij}}, \qquad d\varkappa_{ij} = d\lambda_2 \frac{\partial f_2}{\partial s_{ij}} \tag{2.3}$$

where $d\lambda_1$ and $d\lambda_2$ are certain proportionality factors.

We note that it would have been possible to start from the expression for the work $dA_{12} = (\sigma_{ij} - s_{ij})de_{ij}$. The final relations would have been obtained unaltered, since $\partial(\sigma_{ij} - s_{ij})/\partial\sigma_{ij} = 1$ and, consequently,

$$\frac{\partial f_1}{\partial \sigma_{ij}} = \frac{\partial f_1}{\partial (\sigma_{ij} - s_{ij})}$$

Using the assumption concerning the nature of microstresses, we obtain

$$d\,(e_{ij}' - \varkappa_{ij}') = cds_{ij} \tag{2.4}$$

where c may be considered as a function of the invariants of the tensor s_{ij} and even the tensors σ_{ij}, $\sigma_{ij} - s_{ij}$. Generally speaking, it is necessary to add also the condition

$$e - \varkappa = Ks \tag{2.5}$$

where e, κ, s are the first invariants of the respective tensors, and the quantity K may be considered as a function of the invariants of the stress tensors.

If the functions f_1 and f_2 do not depend on the first invariants of the tensors $\sigma_{ij} - s_{ij}$ and s_{ij} respectively, then it follows from (2.3) that $e_{ij} = e_{ij}'$, $\kappa_{ij} = \kappa_{ij}'$, $e = \kappa = 0$. Since $s \neq 0$, then $K = \infty$.

3. We shall consider now the case of plane strain. We assume that the flow conditions (2.1) do not depend on the third invariants of the deviators of stresses σ_{ij} and s_{ij}.

Taking $e_z = \kappa_z = 0$, we obtain the flow conditions

$$[(\sigma_x - s_x) - (\sigma_y - s_y)]^2 + 4\,(\tau_{xy} - s_{xy})^2 = 4k_1^2 \tag{3.1}$$

$$(s_x - s_y)^2 + 4s_{xy}^2 = 4k_2^2 \tag{3.2}$$

The flow rule can be written in the form

$$\frac{de_x}{(\sigma_x - s_x) - (\sigma_y - s_y)} = \frac{de_y}{(\sigma_y - s_y) - (\sigma_x - s_x)} = \frac{de_{xy}}{2\,(\tau_{xy} - s_{xy})} \tag{3.3}$$

$$\frac{d\varkappa_x}{s_x - s_y} - \frac{dx_y}{s_y - s_x} - \frac{d\varkappa_{xy}}{2s_{xy}} \tag{3.4}$$

We rewrite the conditions (2.4) in the form

$$d\,(e_x - \varkappa_x) = \frac{c}{2}\,d\,(s_x - s_y), \qquad d\,(e_y - \varkappa_y) = \frac{c}{2}\,(s_y - s_x)$$

$$d\,(e_{xy} - \varkappa_{xy}) = cds_{xy} \tag{3.5}$$

Considering, for simplicity, that c = const and assuming that in the initial state all the components of stress and strain are equal to zero, we obtain from (3.5)

$$e_x - \varkappa_x = \frac{c}{2}(s_x - s_y), \quad e_y - \varkappa_y = \frac{c}{2}(s_y - s_x), \quad e_{xy} - \varkappa_{xy} = cs_{xy} \tag{3.6}$$

The conditions (3.1) and (3.2) will be satisfied with

$$\begin{aligned} \sigma_x &= \omega + k_1 \cos 2\theta + s_x, & s_x &= s + k_2 \cos 2\psi \\ \sigma_y &= \omega - k_1 \cos 2\theta + s_y, & s_y &= s - k_2 \cos 2\psi \\ \tau_{xy} &= k_1 \sin 2\theta + s_{xy}, & s_{xy} &= k_2 \sin 2\psi \end{aligned} \tag{3.7}$$

It follows from (3.7)

$$\sigma_x = \sigma + k_1 \cos 2\theta + k_2 \cos 2\psi, \qquad \sigma_y = \sigma - k_1 \cos 2\theta - k_2 \cos 2\psi$$

$$\tau_{xy} = k_1 \sin 2\theta + k_2 \sin 2\psi \qquad (\sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) = \omega + s) \tag{3.8}$$

The conditions (3.3) and (3.4) assume the form

$$de_x \sin 2\theta - de_{xy} \cos 2\theta = 0, \qquad de_x + de_y = 0 \tag{3.9}$$

$$d\varkappa_x \sin 2\psi - d\varkappa_{xy} \cos 2\psi = 0, \qquad d\varkappa_x + d\varkappa_y = 0 \tag{3.10}$$

From (3.5) we obtain

$$d(e_x - \varkappa_x) = -2ck_2 \sin 2\psi d\psi, \qquad d(e_{xy} - \varkappa_{xy}) = -2ck_2 \cos 2\psi d\psi \tag{3.11}$$

Eliminating the quantity $d\kappa_{ij}$ from (3.10) and (3.11), we find

$$de_x \sin 2\psi - de_{xy} \cos 2\psi = -2ck_2 d\psi \tag{3.12}$$

In all the relations derived above, the expressions de_{ij} and $d\psi$ determine the increments of the corresponding components depending on a time parameter. Dividing these expressions by dt, we can obtain the rates of the corresponding quantities. In the following, we shall use the Eulerian description. We denote by ε_{ij} the tensor of deformation-rate and by u and v the components of velocity along the axes x and y, respectively. We note that

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x} u + \frac{\partial\psi}{\partial y} v$$

Substituting the Expressions (3.7) into the equations of equilibrium, we obtain

$$\begin{aligned}&\frac{\partial\sigma}{\partial x}-2k_1\sin 2\theta\frac{\partial\theta}{\partial x}+2k_1\cos 2\theta\frac{\partial\theta}{\partial y}-2k_2\sin 2\psi\frac{\partial\psi}{\partial x}+2k_2\cos 2\psi\frac{\partial\psi}{\partial y}=0\\&\frac{\partial\sigma}{\partial y}+2k_1\cos 2\theta\frac{\partial\theta}{\partial x}+2k_1\sin 2\theta\frac{\partial\theta}{\partial y}+2k_2\cos 2\psi\frac{\partial\psi}{\partial x}+2k_2\sin 2\psi\frac{\partial\psi}{\partial y}=0\end{aligned}\tag{3.13}$$

It is necessary to add also Equations (3.9)

$$\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\sin 2\theta-\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos 2\theta=0,\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\tag{3.14}$$

and, finally, Equation (3.12)

$$\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\sin 2\psi-\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos 2\psi=-4ck_2\frac{d\psi}{dt}\tag{3.15}$$

We note that the system of five equations (3.13) - (3.15) contains five unknowns: σ, θ, ψ, u, v. Denoting by $\chi(x, y, t) = 0$ the equation of the characteristic surface, we obtain the characteristic determinant of this system of equations in the form

$$\frac{d\chi}{dt}(\chi_x^2\cos 2\theta+2\chi_x\chi_y\sin 2\theta\cos 2\theta-\chi_y^2\cos 2\theta)=0\tag{3.16}$$

$$\left(\chi_x=\frac{\partial\chi}{\partial x},\quad \chi_y=\frac{\partial\chi}{\partial y}\right)$$

Consequently, the system (3.13) - (3.15) is always of the hyperbolic type. In the plane xy, we have the orthogonal characteristics

$$\left(\frac{dy}{dx}\right)_{1,2}=\tan\left(\theta\pm\frac{\pi}{4}\right)\tag{3.17}$$

The relations (3.13) assume the following form along the characteristics

$$\begin{aligned}&\frac{\partial\sigma}{\partial\xi}-2k_1\frac{\partial\theta}{\partial\xi}-2k_2\left[\frac{\partial\psi}{\partial\xi}\cos 2(\psi-\theta)+\frac{\partial\psi}{\partial\eta}\sin 2(\psi-\theta)\right]=0\\&\frac{\partial\sigma}{\partial\eta}+2k_1\frac{\partial\theta}{\partial\eta}+2k_2\left[\frac{\partial\psi}{\partial\xi}\sin 2(\psi-\theta)+\frac{\partial\psi}{\partial\eta}\cos 2(\psi-\theta)\right]=0\end{aligned}\tag{3.18}$$

where $d\xi$ and $d\eta$ denote arc elements along the characteristics.

The relations (3.14) imply that along the characteristics the Geiringer relations are valid [4]

$$dU-V\,d\theta=0,\qquad dV+U\,d\theta=0\tag{3.19}$$

where U and V denote the components of velocity along the characteristics.

We shall indicate a series of particular cases. If the yield point and the flow point coincide, $k_1 = m$ ($k_2 = 0$), then (3.14) and (3.15) imply $\theta = \psi$, and in this case the usual relations of perfect plasticity are valid.

For $d\psi/dt = 0$, Equation (3.15) again indicates that $\theta = \psi$, and from (3.8) we see that the usual relations of perfect plasticity are valid, with the yield stress $m = k_1 + k_2$. It is interesting to note that $d\psi/dt = 0$ only for $\theta = \psi$, and if $\theta \neq \psi$ then it is always $d\psi/dt \neq 0$. The quantity ψ, in general, tends to coincide with θ, and the closer ψ approaches θ the smaller is $d\psi/dt$. This fact was illustrated previously by the use of the kinematical model.

If the connections between the elements with solid friction are rigid, i.e. $c = 0$, then Equation (3.5) shows that $e_{ij} = \kappa_{ij}$. From (3.14) and (3.15) we again obtain $\theta = \psi$ and the basic relations reduce to those for perfectly plastic solids with the yield stress $m = k_1 + k_2$.

Finally, if $k_1 = 0$, then $\sigma_{ij} = s_{ij}$, and the relations of an ideal elastic-plastic solid are valid.

Let us note certain characteristic aspects of perfectly plastic flow in the presence of residual microstresses. As in the case of non-existence of microstresses, characteristics form an orthogonal system, but the Hencky theorems [4] do not hold. The maximum shear stresses τ_{max} occur not along characteristics. The lines of discontinuity of velocities, according to (3.14), coincide with characteristics, as it is in the theory of perfect plasticity without microstresses.

It is easy to formulate the basic boundary value problems of Goursat, of Cauchy, and the mixed one, and to outline numerical methods of solution. But in this case, a state of deformation is reached through a region of hardening; therefore, it is necessary to keep in sight the fact that the final solution depends on the history of loading.

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BIBLIOGRAPHY

1. **Kadashevich, Iu.I. and Novozhilov, V.V., Teoriia plastichnosti, uchityvaiushchaia mikronapriazheniia (Theory of plasticity taking into account microstresses).** ***PMM*** **Vol. 22, No. 1, 1958.**

2. **Prager, V., Vliianie deformatsii na uslovie plastichnosti viazko-plasticheskikh tel (Influence of deformation on the yield condition in visco-plastic bodies).** *Sb. Teoriia plastichnosti.* **Moscow, IIL, 1948.**

3. Ishlinskii, A.Iu., Obshchaia teoriia plastichnosti s lineinym uprochneniem (General theory of plasticity with linear hardening). *Ukr. mat. zh.* Vol. 6, No. 3, 1954.

4. Prager, W. and Hodge, P.G., *Teoriia ideal'no plasticheskikh tel* (Russian translation of *Theory of Perfectly Plastic Solids*). Moscow, IIL, 1956.

5. Prager, W., *Teoriia plastichnosti: obzor sovremennykh issledovanii v knige Pragera, W. i Hodzha, F. Teoriia ideal'no plasticheskikh tel (Theory of Plasticity: Review of Contemporary Research in the book by Prager and Hodge, Theory of Perfectly Plastic Solids)*. Moscow, IIL, 1956.

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